

Fluctuation of heat current in Josephson junctions

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We discuss the statistics of heat current between two superconductors at different temperatures connected by a generic weak link. As the electronic heat in superconductors is carried by Bogoliubov quasiparticles, the heat transport fluctuations follow the Levitov–Lesovik relation. We identify the energy-dependent quasiparticle transmission probabilities and discuss the resulting probability density and fluctuation relations of the heat current. We consider multichannel junctions, and find that heat transport in diffusive junctions is unique in that its statistics is independent of the phase difference between the superconductors. Curiously, phase dependence reappears if phase coherence is partially broken.

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I. INTRODUCTION

Heat transport through junctions between superconductors is significantly affected by superconductivity.¹ In tunnel Josephson junctions, superconducting phase coherence manifests as a component of the thermal conductance that oscillates with the phase difference between the superconductors,^{2–5} a prediction which was confirmed by recent experiments.^{6,7} In general, both the sign and the magnitude of the oscillations depend on the transparency of the junction in question.⁴

Previous studies have largely concentrated on the ensemble average value of the heat currents. However, in reality the heat current driven by a temperature difference through a superconducting junction is not constant in time, but fluctuates. When mesoscopic systems are considered, this can lead to fluctuations in other quantities — such as the energy stored on a small metal island — and eventually, in measurable observables, such as charge current^{8–10} or temperature measured by a generic temperature probe¹¹. In addition to the theoretical question on how the coherent physics of phase differences in superconducting order parameters manifest in statistical properties of heat transport, questions on fluctuations can also be of interest in systems that utilize mesoscopic superconductors in a nonequilibrium settings for example for radiation detection¹².

In this work, we find the full statistics of temperature-driven heat transport in a Josephson junction of arbitrary transparency, as illustrated in Fig. 1. Previously, the energy transport statistics has been discussed in the tunneling limit,¹³ and we recover these results as a special case. We separate the energy transport to elementary quasiparticle transport events. In agreement with the Levitov–Lesovik formula,^{14,15} the elementary quasiparticle transmission probabilities are governed by the total transmission eigenvalues $T_{n\pm}(E)$ [Eq. (13)] of the Bogoliubov–de Gennes scattering problem of the interface. We compute the heat current noise and discuss its parameter de-

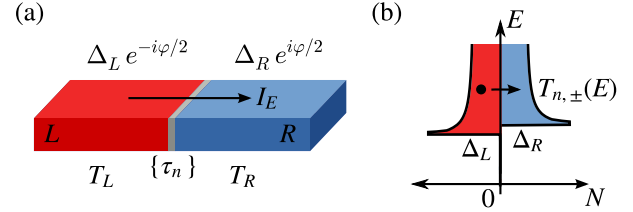


FIG. 1: (a) Heat current I_E flows from superconductor L to another superconductor R coupled to it, driven by a difference in the temperatures $T_L > T_R$. The heat current is modulated by the phase difference φ between the order parameters Δ_L, Δ_R of the superconductors. The Josephson junction connecting the two is described by the set of (spin-independent) transmission eigenvalues $\{\tau_n\}$ of the normal-state scattering matrix of the interface. (b) The excitations that carry heat in superconductors are Bogoliubov quasiparticles, whose density of states $N(E)$ is illustrated. Heat current and its fluctuation statistics is fully determined by their energy-dependent transmission probabilities $T_{n\pm}(E)$. Each normal-state quantum channel (τ_n) splits into inequivalent particle-hole transmission channels (\pm) due to superconductivity.

pendence in single-channel junctions. We derive results corresponding to dirty-interface¹⁶ and diffusive¹⁷ multichannel junctions. The heat statistics in the diffusive limit turns out to have no phase dependence, except in the presence of inelastic effects.

II. MODEL

We consider heat transport between two superconducting terminals that are connected by a generic contact described by the transmission eigenvalues $\{\tau_j\}$. The terminals are assumed to lie at different temperatures. We make use of the Keldysh–Nambu Green function formulation for transport in superconducting structures,^{18–20} and the quasiclassical boundary condition description of

a weak link between bulk superconductors in the diffusive limit.^{21,22}

At equilibrium, electrons inside a superconducting terminal at temperature T with superconducting gap Δ are described by the quasiclassical equilibrium Green function $\check{g}_0(E)$,

$$\check{g}_0 = \begin{pmatrix} \hat{g}_0^R & (\hat{g}_0^R - \hat{g}_0^A)h_0 \\ 0 & \hat{g}_0^A \end{pmatrix}, \quad (1)$$

$$\hat{g}^R = \begin{pmatrix} \frac{E}{\sqrt{E^2 - |\Delta|^2}} & \frac{\Delta}{\sqrt{E^2 - |\Delta|^2}} \\ -\frac{\Delta^*}{\sqrt{E^2 - |\Delta|^2}} & -\frac{E}{\sqrt{E^2 - |\Delta|^2}} \end{pmatrix}, \quad (2)$$

and $\hat{g}^A = -\hat{\tau}_3(\hat{g}^R)^\dagger\hat{\tau}_3$, where $\hat{\tau}_3$ is the third spin matrix in the Nambu space. Temperature enters in the equilibrium distribution function $h_0 = \tanh \frac{E}{2T}$. Presence of sub-gap states in superconductors can be taken into account via a Dynes parameter, by replacing $E \mapsto E \pm i\Gamma/2$ in $g^{R/A}$, where Γ is a relaxation rate due to e.g. electron-phonon or other interactions.

Statistics of heat flow can be conveniently described via the two-point generating function^{23,24}

$$W_\alpha(u, t) = \text{Tr}[e^{iuH_\alpha} U(t) e^{-iuH_\alpha} \rho(0) U(t)^\dagger], \quad (3)$$

where $\alpha = L, R$ indicates the terminal whose internal energy is counted, and H_α are the BCS Hamiltonians of the superconducting terminals. These functions can be computed using the Keldysh approach of Refs. 23,25,26, as follows. Differentiating Eq. (3) we obtain,

$$\partial_u W(u, t) = i\langle H_\alpha(t) - H_\alpha(0) \rangle_u W(u, t), \quad (4)$$

where $\langle X \rangle_u = \text{Tr}[X U_+ \rho(0) U_-^\dagger] / \text{Tr}[U_+ \rho(0) U_-^\dagger]$ is an expectation value computed with modified time evolution operators $U_\pm = e^{\pm iuH_\alpha/2} U e^{\mp iuH_\alpha/2}$ including the counting field with differing signs on different Keldysh branches. This results only to time shifts in the interaction picture Green function of lead α , as the energy counting factor has the same form as time evolution.²³ Transforming to the Green function representation²⁷ used above, the time shifts are represented by

$$\check{g}_\alpha(E, u) = e^{iuE\check{\sigma}_1/2} \check{g}_\alpha(E) e^{-iuE\check{\sigma}_1/2}. \quad (5)$$

Computing the expectation value in Eq. (4) via the quasiclassical boundary condition approach,^{21,22} and integrating in u results to the well-known action of superconducting contacts^{23,25,26,28}

$$\ln W_R(u, t) = \frac{1}{2} \sum_n \text{Tr} \ln \left[1 + \frac{\tau_n}{4} ([\check{g}_L, \check{g}_R(u)]_+ - 2) \right] + C, \quad (6)$$

where C is a normalization constant, and Tr includes energy integration in addition to Keldysh-Nambu matrix trace.

III. GENERATING FUNCTION

An important difference in energy transport compared to charge statistics follows from the fact that an Andreev reflection does not transfer energy. In the present formulation, this is visible in the fact that $(\Gamma \rightarrow 0^+)$

$$\check{g}(E, u) = \hat{g}^R(E) \otimes \begin{cases} 1, & |E| < \Delta, \\ e^{\frac{iuE\check{\sigma}_1}{2}} \begin{pmatrix} 1 & 2h \\ 0 & -1 \end{pmatrix} e^{-\frac{iuE\check{\sigma}_1}{2}}, & |E| > \Delta. \end{cases} \quad (7)$$

There is no energy transfer at sub-gap energies, where there are no quasiparticles, assuming no broadening in the spectrum of the superconductors.

The generating function can be found by direct substitutions into Eq. (6). It is however useful to make use of $\check{g}_{L/R}^2 = 1$ and rewrite

$$1 + \frac{\tau_n}{4} ([\check{g}_L, \check{g}_R(u)]_+ - 2) = \frac{[q_n + \check{g}_L \check{g}_R(u)][q_n + \check{g}_R(u) \check{g}_L]}{(1 + q_n)^2}, \quad (8)$$

where $q_n = -1 + 2/\tau_n + 2\sqrt{1 - \tau_n}/\tau_n$ are the eigenvalues of the hermitian square of the corresponding transfer matrix,^{25,29} so that

$$\ln W_R(u) = \sum_n \ln \det[q_n + \check{g}_L \check{g}_R(u)] + C', \quad (9)$$

where C' is a normalization constant.

The product structure $\check{g} = \hat{g}^R \otimes \check{V}$ of Eq. (7) implies that the Keldysh and Nambu components can be diagonalized separately. This yields

$$\ln W_R(u) = 2t_0 \int_{\max(\Delta_L, \Delta_R)}^\infty \frac{dE}{2\pi} \sum_n \sum_{\alpha\beta=\pm 1} \ln \frac{\mu^\alpha + q_n \lambda^\beta}{1 + q_n \lambda^\beta}, \quad (10)$$

where t_0 is the measurement time, and $\{\lambda, 1/\lambda\}$ and $\{\mu, 1/\mu\}$ are the eigenvalues of $\hat{g}_L^R \hat{g}_R^R$ and $\check{V}_L \check{V}_R(u)$, respectively.

Solving the eigenvalue problems, Eq. (10) can be rewritten in the form of a characteristic function of a multinomial distribution:

$$\ln W_R(u) = 2t_0 \int_0^\infty \frac{dE}{2\pi} \sum_n \sum_{\beta=\pm 1} \ln \sum_k e^{ikuE} p_{k,n,\beta}(E), \quad (11)$$

with the event probabilities

$$p_{k,n,\beta}(E) = \begin{cases} T_{n\beta}(E)[1 - f_R(E)]f_L(E), & k = +1, \\ T_{n\beta}(E)[1 - f_L(E)]f_R(E), & k = -1, \\ 1 - p_{+1,n,\beta}(E) - p_{-1,n,\beta}(E), & k = 0, \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

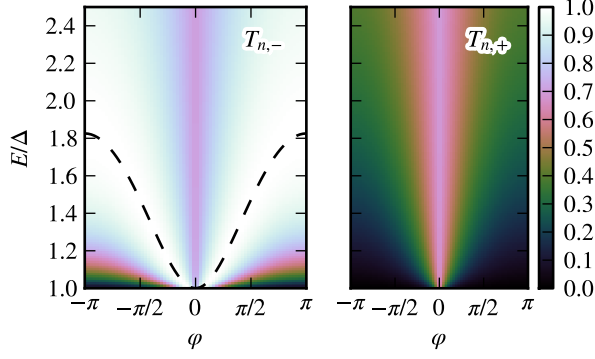


FIG. 2: Transmission eigenvalues $T_{n,\pm}(E)$ calculated for $\tau_n = 0.7$ and $\Delta_L = \Delta_R = \Delta$. Location of the resonance (15) is indicated with a dashed line. At $\varphi = 0$, $T_{n,\pm} = \tau_n$.

Here, $f_{L/R}$ are the electron Fermi distribution functions in the left and right terminals, and

$$T_{n\beta}(E) = \theta(E^2 - |\Delta_L|^2) \theta(E^2 - |\Delta_R|^2) \frac{4\lambda^\beta q_n}{(1 + \lambda^\beta q_n)^2}, \quad (13a)$$

$$\lambda = \exp \operatorname{arccosh} \frac{E^2 - |\Delta_L| |\Delta_R| \cos \varphi}{\sqrt{E^2 - |\Delta_L|^2} \sqrt{E^2 - |\Delta_R|^2}}. \quad (13b)$$

Here, φ is the phase difference between the order parameters of the two superconductors. The generating function describes events where a quasiparticle at energy $E > \Delta$ attempts to move from the left to the right ($k = +1$) or from the right to the left ($k = -1$). The probability that such a transmission event succeeds depends both on the transparency of the transmission channel (q_n , τ_n), and on superconductivity (λ).

The probabilities $T_{n,\pm}$ are the transmission eigenvalues of a Bogoliubov–de Gennes (BdG) scattering problem. The corresponding scattering matrix (for $\Delta_L = \Delta_R$) can be found in Ref. 30. Direct evaluation of the transmission

eigenvalues using the results there gives

$$\operatorname{eig} tt^\dagger = \{T_{n,\pm}\}, \quad (14)$$

which coincides with Eq. (13) above. A similar connection can be made to well-known results in N/S junctions,³¹ ($\Delta_L = 0$, $\Delta_R = \Delta > 0$), after identifying the barrier reflectivity in Ref. 31 as $Z = (q - 1)/\sqrt{4q}$. That the energy statistics is related to these scattering matrices stems from the fact that the counting statistics of Bogoliubov quasiparticles, which carry all of the electronic energy current, must follow the well-known Levitov–Lesovik result^{14,15}.

Superconductivity has a significant impact on the transmission eigenvalues. In the normal state, they are simply $T_{n\beta} = 4q_n/(1 + q_n)^2 = \tau_n$ independent of the particle-hole channel index $\beta = \pm 1$. In the superconducting state, they deviate significantly from this, as illustrated in Fig. 2. In particular, there is a transmission resonance in one of the channels:

$$T_{n,-}(E_{\text{res}}) = 1, \quad E_{\text{res}} = \pm \Delta \sqrt{1 + \frac{\tau_n}{1 - \tau_n} \sin^2 \frac{\varphi}{2}}. \quad (15)$$

The above result applies for $\Delta_L = \Delta_R$, but the resonance appears also for $\Delta_L \neq \Delta_R$. This has a large effect especially for junctions whose normal-state transparency is small, in which a large part of the total heat current is carried by the resonance.⁴

A. Breaking phase coherence

We can extend the above results to include broadening of the density of states in the superconductors, by adding a Dynes parameter $\Gamma > 0$. Also in this case, the final generating function obtains the form (11). As the factorization (7) does not apply, the expressions for the transmission eigenvalues $T_{n\beta}(E)$ need to be extracted from the determinant in Eq. (9).

Straightforward calculation yields for $T_{n,\pm}(E)$ the indirect expressions

$$T_{n,+} T_{n,-} = \frac{16q_n^2 \cos^2(\operatorname{Im} \theta_L) \cos^2(\operatorname{Im} \theta_R)}{|1 + q_n^2 + 2q_n \cosh \theta_L \cosh \theta_R - 2q_n \sinh \theta_L \sinh \theta_R \cos \varphi|^2}, \quad (16a)$$

$$\frac{1}{T_{n,+}} + \frac{1}{T_{n,-}} = 1 + \tan(\operatorname{Im} \theta_L) \tan(\operatorname{Im} \theta_R) \cos \varphi + \frac{(q_n^2 + 1) [\cosh(\operatorname{Re} \theta_L) \cosh(\operatorname{Re} \theta_R) - \sinh(\operatorname{Re} \theta_L) \sinh(\operatorname{Re} \theta_R) \cos \varphi]}{2q_n \cos(\operatorname{Im} \theta_L) \cos(\operatorname{Im} \theta_R)}, \quad (16b)$$

where $\theta_{L/R} = \operatorname{arctanh} \frac{\Delta_{L/R}}{E + i(\Gamma/2)}$. The above formulas reduce to Eq. (13) for $\Gamma \rightarrow 0^+$. For $\Gamma > 0$, the results cannot however be written in terms of a single λ as in Eq. (13).

The eigenvalues $T_{n,\pm}(E)$ are plotted for $\Gamma > 0$ in Fig. 3. In general, a finite Γ induces a small number of states inside the gaps of the superconductors, so that

the junction is transparent for heat transport also at sub-gap energies. Similarly as for above-gap transport, the transmission through the two channels \pm can differ sig-

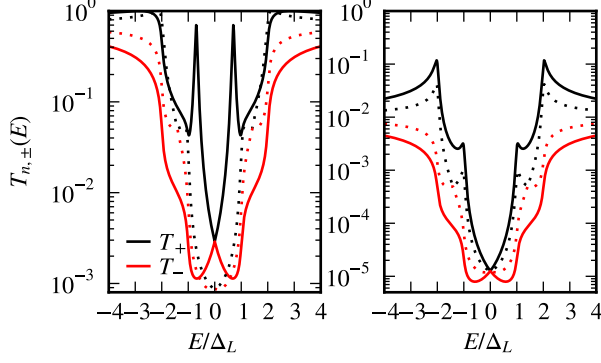


FIG. 3: Transmission eigenvalues $T_{n,\pm}(E)$ at $\varphi = \pi$ (solid) and $\varphi = 0$ (dotted) for $\Gamma = 0.1\Delta$, and $\tau_n = 0.7$ (left panel) or $\tau_n = 0.01$ (right panel). Here, $\Delta_R = 2\Delta_L$.

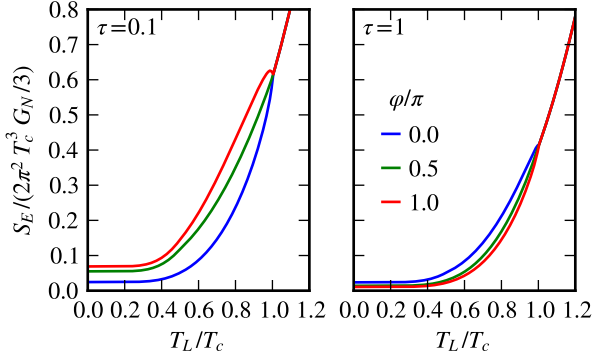


FIG. 4: Heat current noise S_E as a function of temperature T_L , for different values of φ and τ_n (single channel), normalized to its equilibrium normal-state value at $T = T_c$. The temperature $T_R = T_c/2$ is kept fixed. Here, T_c is the BCS critical temperature, and we take the temperature dependence of the energy gaps $\Delta_L(T_L)$, $\Delta_R(T_R)$ into account via the BCS relation, taking $\Delta_L(0) = \Delta_R(0)$.

nificantly. Moreover, we can observe that sharp sub-gap resonances appear in one of the two channels — these are associated with the Andreev bound states and are most prominent in transparent junctions. Moreover, we note that the phase dependence of sub-gap heat transport in high-transparency channels (left panel) can be substantial, whereas it is not as prominent at low transparencies (right panel).

IV. HEAT FLOW STATISTICS

Let us now compute the first moments of the heat statistics. Direct differentiation of Eq. (11) with respect

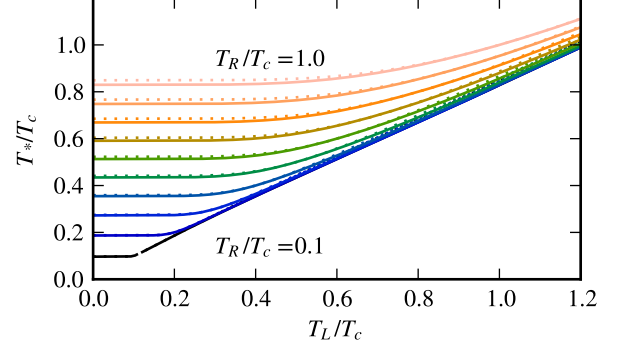


FIG. 5: Temperature $T_*(T_L, T_R)$ at which $S_E(T_L, T_R) = 2T_*^2 G_{th}(T_*)$. T_R sweeps from $0.1T_c$ to T_c in steps of $0.1T_c$. Shown for a single-channel system with $\tau = 0.1$, $\varphi = \pi$ (solid) and $\tau = 0.9$, $\varphi = \pi/4$ (dashed).

to u yields the average heat current,

$$I_E = \sum_n \int_{-\infty}^{\infty} \frac{dE}{2\pi} E [T_{n,+}(E) + T_{n,-}(E)] [f_L(E) - f_R(E)], \quad (17)$$

$$\sum_{\pm} T_{n\pm} = \frac{2\tau_n(E^2 - \Delta^2)[E^2 - \Delta^2 \cos^2 \frac{\varphi}{2} + r_n \Delta^2 \sin^2 \frac{\varphi}{2}]}{(E^2 - \Delta^2(1 - \tau_n \sin^2 \frac{\varphi}{2}))^2}, \quad (18)$$

where $r_n = 1 - \tau_n$. Naturally, this result coincides exactly with that found in Ref. 4. Note, however, that the coefficients \mathcal{D}_{ee} , \mathcal{D}_{he} defined in Ref. 4 do not coincide with $T_{n\pm}$, even though the sums do.

Taking the second derivative, we find the heat current zero-frequency noise

$$S_E = \sum_n \sum_{\beta=\pm 1} \int_{-\infty}^{\infty} \frac{dE}{2\pi} E^2 T_{n\beta} \left\{ f_L(1 - f_R) + f_R(1 - f_L) + T_{n\beta}(f_L - f_R)^2 \right\}. \quad (19)$$

In the tunneling limit, this result coincides with that found in Ref. 13. The behavior of the heat current noise away from equilibrium is shown in Fig. 4. We emphasize the difference of S_E between the opaque (left panel) and transparent junction limit (right panel). In particular, in the former case, the heat current noise is minimized for $\varphi = 0$ whereas in the latter junction S_E is minimized for $\varphi = \pi$. This behavior resembles the one of the thermal conductance of a temperature-biased Josephson weak-link, as predicted in Refs. 4,5.

At equilibrium ($T_L = T_R = T$), the heat current noise obeys the well-known fluctuation relation $S_E = 2G_{th}T^2$ that connects it to the thermal conductance $G_{th} = dI_E/dT_R|_{T_L=T_R}$. Conversely, we can define an effective temperature $T_*(T_L, T_R, \{\tau_n\}, \varphi)$ such that $S_E = 2T_*^2 G_{th}(T_*)$. Such a temperature plot is shown

in Fig. 5. The result is fairly insensitive to the values of τ_n and φ , and the results fall nearly on the same curves for different values of these parameters. At low temperatures, the result converges towards $T_* = \max(T_L, T_R)$.

More generally, the generating function obeys the fluctuation relation²⁴

$$W_R(u) = W_R(-u + iT_L^{-1} - iT_R^{-1}). \quad (20)$$

For the probability distribution of transferring energy ε out of terminal R in time t_0 this implies

$$\frac{P_R(\varepsilon, t_0)}{P_R(-\varepsilon, t_0)} = e^{\varepsilon/T_R} e^{-\varepsilon/T_L}, \quad t_0 \rightarrow \infty. \quad (21)$$

This fluctuation relation is independent of the channel transmissions $T_{n\beta}$, and therefore does not contain information about superconductivity.

A. Diffusive junctions

It is possible to average the above generating functions over known distributions of transmission eigenvalues $\{\tau_n\}$, to obtain results for certain types of multichannel junctions. Let us in particular consider the transmission eigenvalue distribution of a short diffusive junction,¹⁷

$$\sum_{\tau_n} = \int_0^1 d\tau \rho(\tau), \quad \rho(\tau) = \frac{1}{\tau\sqrt{1-\tau}}. \quad (22)$$

Changing the integration variable to q for $\beta = +$ and q^{-1} for $\beta = -$ in Eq. (11):

$$\begin{aligned} \ln W_R(u) = 2t_0 \int_0^\infty \frac{dE}{2\pi} \int_0^\infty \frac{dq}{q} \tau(q) \sqrt{1-\tau(q)} \rho(\tau(q)) \\ \times \ln[1 + \frac{4\lambda q}{(1+\lambda q)^2} F(E, u)], \end{aligned} \quad (23)$$

where $F(E, u) = (e^{iuE} - 1)f_L(1 - f_R) + (e^{-iuE} - 1)f_R(1 - f_L)$. The diffusive distribution (22) cancels the $\tau(q)$ dependent prefactors. The integral can then be evaluated:

$$\ln W_R(u) = 2t_0 \int_{\max(\Delta_L, \Delta_R)}^\infty \frac{dE}{2\pi} 4 \operatorname{arcsinh}^2 \sqrt{F(E, u)}. \quad (24)$$

Note that the result does not depend on λ . The heat transport statistics of a diffusive junction is *independent* of the phase difference between the superconductors. Moreover, the only difference to the normal-state result is the presence of a gap $\max(|\Delta_L|, |\Delta_R|)$ in the energy integration.

The absence of phase oscillations arises as coherent contributions from different channels cancel each other. The sign of the phase oscillation is different for large and small τ_n , and diffusive junctions contain the exact

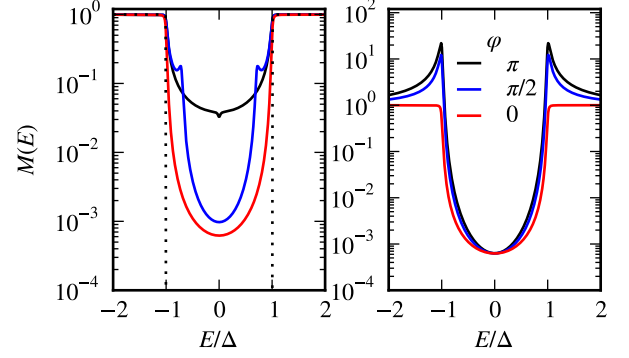


FIG. 6: Heat transparency of a short diffusive junction (left panel) and tunnel junction channel $\tau = 0.01$ (right panel), for inelastic Dynes parameter $\Gamma = 0.05\Delta$. The diffusive-limit phase-independent result for $\Gamma \rightarrow 0^+$ is shown as a dotted line in the left panel.

balance of low and high-transparency channels necessary for the sum to cancel. The diffusive limit distribution is unique in the sense that $\log q_n$ are uniformly distributed²⁹, which is crucial for the above result.

Moreover, it should be noted that the phase dependence reappears in the diffusive limit if inelastic scattering is present ($\Gamma > 0$). The technical reason for this is that it is only in the limit $\Gamma \rightarrow 0$ that superconductivity appears solely as a scale factor λ in the square transfer matrix eigenvalues q .

We can illustrate the reappearance of phase oscillations via the normalized value $M(E) = \sum_{n,\pm} T_{n,\pm}(E) / [2 \sum_n \tau_n]$ that can be understood as a transparency for the heat current (17). The result is shown in Fig. 6. At sub-gap energies, the emergence of the well-known diffusive junction minigap of size $E_g = |\Delta| |\cos \frac{\varphi}{2}|$ is evident. Moreover, as pointed out above for single transparent channels, in stark contrast with low-transparency tunnel junctions, the relative sub-gap change in heat conductivity can be several orders of magnitude in transparent junctions (left panel).

B. Dirty interfaces

A second universal, potentially experimentally interesting eigenvalue distribution is the “dirty interface” distribution^{16,32}

$$\rho(\tau) = \frac{g}{\pi} \frac{1}{\tau^{3/2} \sqrt{1-\tau}}, \quad g = \sum_n \tau_n. \quad (25)$$

Experimentally, this has been found to match results existing in high-transparency tunnel junctions.³³

The generating function can be evaluated also in this

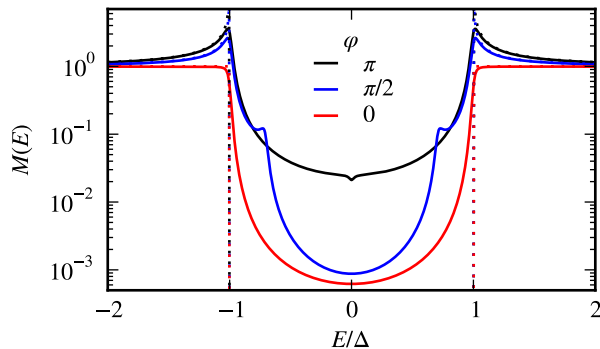


FIG. 7: Heat transparency $M(E) = \sum_{n,\pm} T_{n,\pm}(E)/[2\sum_n \tau_n]$ for the dirty interface transmission distribution. Shown for $\Gamma = 0.05\Delta$ (solid) and $\Gamma \rightarrow 0^+$ (dashed).

case, starting from Eq. (23):

$$\ln W_R(u) = 2t_0 \int_{\max(\Delta_L, \Delta_R)}^{\infty} \frac{dE}{2\pi} 4\sqrt{2}g \sinh^2 \frac{z(E, u)}{4} \times \sqrt{1 + \frac{E^2 - \Delta_L \Delta_R \cos \varphi}{\sqrt{E^2 - \Delta_L^2} \sqrt{E^2 - \Delta_R^2}}}, \quad (26)$$

where $z(E, u) = \text{arccosh}[1 + 2F(E, u)]$. The associated heat transparency entering the heat current is illustrated in Fig. 7. We can note that the above-gap transport resembles that in tunnel junctions, and the sub-gap part that of diffusive junctions. However, in contrast to the tunnel junction result,^{3,4} the energy integral in Eq. (26) is convergent, and does not require cutoffs in the above-gap resonance.

V. DISCUSSION

Electronic transport of heat in Josephson junctions is facilitated by transfer of quasiparticles from one side to the other. Andreev reflections transfer no heat, and so

only quasiparticles contribute. Consequently, the heat transport statistics follows directly from the counting statistics of these excitations. Exactly as in normal-state junctions,^{14,15} this is determined by the transmission eigenvalues of an appropriate scattering problem. In the superconducting state, this statistics is described by Bogoliubov–de Gennes transmission eigenvalues. We find their analytical expressions, Eqs. (13) and (16).

We considered both above-gap and sub-gap transport of heat, the latter by using a toy model for the broadening of the density of states in the superconductors. The general picture that emerges is that sub-gap heat transport is significantly more sensitive to the phase difference in transparent junctions than in tunnel junctions. Interestingly, in diffusive junctions it is in fact only the sub-gap transport that has any phase dependence at all.

The fluctuation statistics of heat current can be measured using similar approaches as previously used for studying the heat currents.^{6,7} For example, one can make the heat capacity of one of the terminals small, and then observe the fluctuation of its total energy via temperature measurements using established experimental techniques¹². Setups utilizing several metal islands^{8,11} could also be a viable approach for probing the statistics experimentally.

In summary, we consider heat current driven by a temperature difference across Josephson junctions of varying transparency. We obtain the generating function of fluctuation statistics in closed form. In addition to describing the fluctuations of heat current, the results provide a way to understand the average heat current in terms of elementary transmission events.

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